Les Houches lecture II.

Coaction for Feynman graphs.

- Will prove a version of Schnetz' coaction conjecture
- Implies a small graphs principle : we do a finite computation at low loop order and it implies a result at all loop orders.

Example : If one can show there is no $\zeta(2)$ in ϕ^4 for graphs with at most 6 edges (I expect this is true)

$\overset{\text{OK}}{\Longrightarrow}$ No $\zeta(n_1,..,n_r)\,\zeta(2)$ ever occurs at any loop order.

- Today massless ϕ^4, but similar ideas should work for any QFT.

I). Set up : motivic periods.

T a (neutral) Tannakian category over $\mathbb{Q}$, with two fiber functors

$$\omega_B, \omega_{dR} : T \longrightarrow \mathrm{Vec}_{\mathbb{Q}} .$$

The ring of motivic periods

$$P^m_T = \mathbb{Q}\langle [M, v, \sigma]^m \big/ \text{equivalence} \sim \;:\; M \in T,\ v \in \omega_{dR}(M),\ \sigma \in \omega_B(M)^\vee \rangle$$

Equivalence : (i) Linear in v and σ :

$$[M, \lambda_1 v_1 + \lambda_2 v_2, \sigma]^m = \lambda_1 [M, v_1, \sigma]^m + \lambda_2 [M, v_2, \sigma]^m$$

idem. for σ

(ii) If $\phi : M \longrightarrow N$ a morphism in T then

$$[M, v, \phi^t(\sigma)]^m \sim [N, \phi(v), \sigma]^m$$

where $\begin{cases} \omega_{dR}(M) \xrightarrow{\phi} \omega_{dR}(N) \\ \omega_B(M)^\vee \xleftarrow{\phi^t} \omega_B(N)^\vee \end{cases}$

Think of "$\int_\sigma v$".

Multiplication : $[M_1, v_1, \sigma_1]^m \, [M_2, v_2, \sigma_2]^m = [M_1 \otimes M_2, \, v_1 \otimes v_2, \, \sigma_1 \otimes \sigma_2]^m$

Suppose we have a functorial isom.

$$\mathrm{comp}_{B,dR} : \omega_{dR}(M) \otimes \mathbb{C} \xrightarrow{\sim} \omega_B(M) \otimes \mathbb{C}$$

Then there exists a period homomorphism

$$\begin{aligned} \mathrm{per} : P^m_T &\longrightarrow \mathbb{C} \\ [M, v, \sigma]^m &\longmapsto \langle \mathrm{comp}_{B,dR}(v), \sigma \rangle \end{aligned}$$

Variant : Ring of de Rham periods :

$$P^{dR}_T = \mathbb{Q}\left\langle [M, v, f]^{dR} \Big/ \text{equivalence} \sim \; : \; \begin{matrix} M \in T, \; v \in \omega_{dR}(M) \\ f \in \omega_{dR}(M)^\vee \end{matrix} \right\rangle$$

Equivalence as before

Coaction formula :

$$\begin{aligned} P^m_T &\longrightarrow P^{dR}_T \otimes P^m_T \\ [M, v, \sigma]^m &\longmapsto \sum_{\{f\}} [M, v, f^\vee]^{dR} \otimes [M, f, \sigma]^m \end{aligned}$$

Sum over elements of basis $\{f\}$ of $\omega_{dR}(M)$; $\{f^\vee\}$ dual basis.

NB : Coaction is equivalent to action of a group

$$G^{dR}_T \circlearrowright P^m_T$$

where $G^{dR}_T = \mathrm{Spec}\, P^{dR}_T$.

Examples

① $T = MT(\mathbb{Z})$ mixed Tate motives over $\mathbb{Z}$. Its motivic periods are called *motivic multiple zeta values*:

$$\zeta^{m}(n_1,\dots,n_r) \qquad \mathrm{per}(\zeta^{m}(n_1,\dots,n_r)) = \zeta(n_1,\dots,n_r)$$

Formula for coaction is known (Ihara, Goncharov + B.)

Warning: de Rham MZVs $\zeta^{dR}(n_1,\dots,n_r)$ have no (obvious) period and satisfy $\zeta^{dR}(2) = 0$. These are the 'motivic MZV's' of Goncharov.

Model: $\mathcal{U}^{dR} = \mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle$ with shuffle product, f_{2n+1} in degree $2n+1$. Underlying vector space has a basis "words in the f_i's".

$$\mathcal{U}^{m} = \mathcal{U}^{dR} \otimes \mathbb{Q}[f_2] \qquad \text{graded module, } f_2 \text{ degree } 2.$$

Coaction:

$$\mathcal{U}^{m} \longrightarrow \mathcal{U}^{dR} \otimes \mathcal{U}^{m}$$

$$f_{i_1}\dots f_{i_r} f_2^{k} \longmapsto \sum_j f_{i_1}\dots f_{i_j} \otimes f_{i_{j+1}}\dots f_{i_r} f_2^{k}$$

Theorem: There is a canonical (depends on choice of Hoffman-Lyndon basis) isomorphism:

$$P^{\bullet}_{MT(\mathbb{Z})} \xrightarrow{\sim} \mathcal{U}^{\bullet} \qquad \bullet = m, dR$$

st respects coactions

$$\begin{array}{ccc} P^{m}_{MT(\mathbb{Z})} & \longrightarrow & P^{dR}_{MT(\mathbb{Z})} \otimes P^{m}_{MT(\mathbb{Z})} \\ \downarrow \wr & & \downarrow \wr \qquad \downarrow \wr \\ \mathcal{U}^{m} & \longrightarrow & \mathcal{U}^{dR} \otimes \mathcal{U}^{m} \end{array}$$

Examples:

$$\zeta^{m}(2n+1) \rightsquigarrow f_{2n+1}$$

$$\zeta^{m}(2)^{k} \rightsquigarrow f_2^{k}$$

$$\zeta^{m}(3,4) \rightsquigarrow 17 f_7 - 10 f_5 f_2$$

$$\zeta^{m}(5,3) \rightsquigarrow -\frac{1117}{324} f_2^{4} + 6 f_3 f_5 - f_5 f_3$$

(2) $MT(\mathbb{Z}[\tfrac{1}{2}])$. Motivic Euler sums $\zeta^m(\pm n_1, .., \pm n_r)$

Period : $$\zeta(\underset{\underset{\varepsilon_1}{\uparrow}}{\pm n_1}, .., \underset{\underset{\varepsilon_r}{\uparrow}}{\pm n_r}) = \sum_{1 \le k_1 < .. < k_r} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_r^{n_r}}{k_1^{n_1} \cdots k_r^{n_r}} \qquad \varepsilon_i \in \{\pm\}$$

Same theorem as above (Deligne) but with

$$\mathcal{U}_2^{dR} = \mathbb{Q}\langle \underset{\sim}{g_2}, f_3, f_5, f_7, \ldots \rangle$$

new generator in degree 1 $\longleftrightarrow$ $\log^m(2)$.

Motivic Euler sums $\longleftrightarrow$ L.C. of words in $\{g_2, f_3, \ldots\} \cdot \zeta_2^k$.

(3) Similar story with 6^{th} roots of 1. Lots of new generators, in particular are in degree 2 f_2 (eg $Li_2(\zeta_6)$).

II. <u>Graph motives.</u>

Take a category H, where objects are triples $(M_{dR}, M_B, comp_{B,dR})$:

(i) $M_{dR} \in Vec_{\mathbb{Q}}$ with increasing filt. $W_\bullet$, dec. filt. $F^\bullet$

(ii) $M_B \in Vec_{\mathbb{Q}}$ with increasing filt. $W_\bullet$

(iii) $comp_{B,dR} : M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} M_B \otimes \mathbb{C}$

st compatible & $(M_B, W_\bullet, F^\bullet \otimes \mathbb{C})$ is a MHS.

<u>Graph motives</u>. G connected graph, N edges, $\psi_G \in \mathbb{Z}[\alpha_e]$ graph polynomial.

- Graph hypersurface

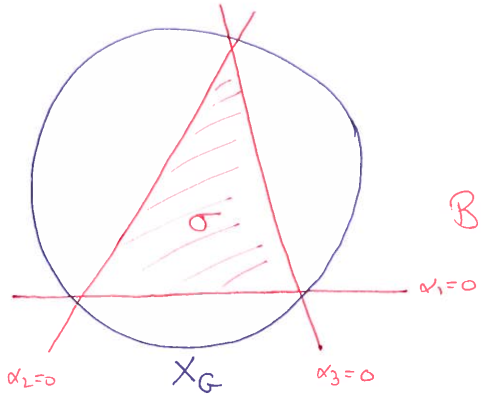
$$X_G = \{\psi_G = 0\} \subseteq \mathbb{P}^{N-1}$$

- Coordinate hyperplanes

$$B = \cup \{\alpha_e = 0\} \subseteq \mathbb{P}^{N-1}$$

Example : $G =$ 

$$\psi_G = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$



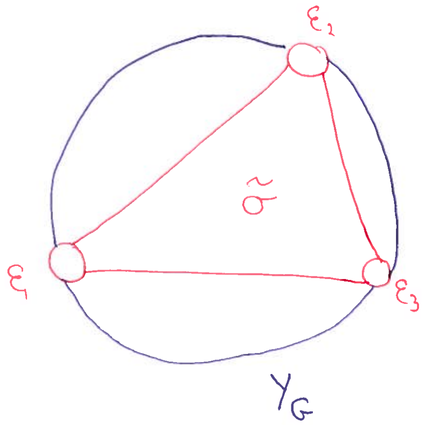
Let $\pi : P_G \longrightarrow \mathbb{P}^{N-1}$ wonderful compactification (eg de Concini – Procesi). Blow up in intersections of coordinate hyperplanes.

Y_G = strict transform of X_G

$\tilde{B}$ = total inverse image of B

$$\mathrm{mot}_G = H^{N-1}\left(P_G \setminus Y_G,\ \tilde{B} \setminus (\tilde{B} \cap Y_G)\right)$$ (Bloch - Esnault - Kreimer)

Claim : $\left((\mathrm{mot}_G)_{dR},\ (\mathrm{mot}_G)_B,\ \mathrm{comp}_{B,dR}\right) \in \mathcal{H}$



Let $\omega_G = \pi^*\left(\dfrac{\Omega_G}{\psi_G^2}\right)$ Feynman integrand

$\tilde{\sigma}$ = closure of strict trans. of σ.

Define : If G primitive divergent, then

$$I^{\mathfrak{M}}_G = \left[\mathrm{mot}_G,\ [\omega_G],\ [\tilde{\sigma}]\right] \in \mathcal{P}^{\mathfrak{M}}_{\mathcal{H}}$$ "motivic amplitude"

[B-E-K ; ω_G has no poles on exceptional locus]

$$\mathrm{per}\left(I^{\mathfrak{M}}_G\right) = I_G$$

$$= \int_\sigma \frac{\Omega_G}{\psi_G^{\,2}}$$ Feynman amplitude

III Schnetz conjecture (reformulated)

Define : $\mathcal{P}^{m}_{\phi^4} = \langle I^{m}_{G} : G \text{ primitive divergent} \rangle_{\mathbb{Q}} \subseteq \mathcal{P}^{m}_{H}$

Conjecture (Schnetz) : $\mathcal{P}^{m}_{\phi^4} \longrightarrow \mathcal{P}^{dR}_{H} \otimes \mathcal{P}^{m}_{\phi^4}$ closed under coaction

Very strong conjecture — has huge predictive power for amplitudes.

Variant ; define

$$\mathcal{P}^{m}_{\tilde{\phi}^4} = \langle [mot_G, \omega, \tilde{\sigma}]^{m} \text{ for any } \omega \in (mot_G)_{dR} \rangle \subseteq \mathcal{P}^{m}_{H}$$

Theorem : Then $\mathcal{P}^{m}_{\tilde{\phi}^4} \longrightarrow \mathcal{P}^{dR}_{H} \otimes \mathcal{P}^{m}_{\tilde{\phi}^4}$ is indeed closed under coaction.

Proof : $$[mot_G, \omega, \tilde{\sigma}]^{m} \longmapsto \sum_{\substack{\{f\} \\ \text{basis of} \\ (mot_G)_{dR}}} \underbrace{[mot_G, \omega, f^{\vee}]^{dR}}_{\substack{??? \\ \text{Mysterious}}} \otimes \underbrace{[mot_G, f, \tilde{\sigma}]^{m}}_{\in \mathcal{P}^{m}_{\tilde{\phi}^4}} \quad [$$

I claim that the theorem, together with small graphs principle (below) is enough to explain most, if not all observed structures.

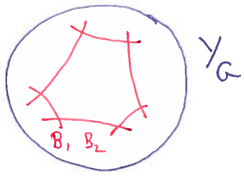
Remark (1): We get $G^{dR}_{H} \circlearrowright \mathcal{P}^{m}_{\tilde{\phi}^4}$. Its action factors through a quotient G_{ϕ^4} which is (Hodge)-motivic Galois gp of ϕ^4.

(2) Motives of graphs with subdivergences were defined in Brown - Kreimer. Different story.

IV. Small graphs principle

Graph motives need to be generalized for subquotients of ϕ^4 graphs.

Write $\tilde{B} = \coprod_i B_i$ n.c.



Exists spectral sequence

$$\mathcal{E}'_{pq} = \bigoplus_{|I|=p} H^q\left(B_I \setminus (Y_G \cap B_I)\right) \qquad B_I = \bigcap_{i \in I} B_i$$

$$\Longrightarrow \quad \mathrm{mot}_G$$

<u>Main point</u>: (Bloch + ?)

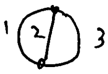
$$B_I \setminus (Y_G \cap B_I) \cong (P_{\gamma_1} \setminus Y_{\gamma_1}) \times (P_{\gamma_2} \setminus Y_{\gamma_2})$$

$\gamma_1 \subseteq G$ subgraph

$\gamma_2 = G/\!/\gamma$ quotient.

Comes from "factorization":

$$\psi_G = \psi_\gamma \, \psi_{G/\!/\gamma} + R_\gamma \qquad \text{remainder}$$

<u>Example</u>: $G =$ 

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = (\alpha_1+\alpha_2)\cdot\alpha_3 + \underbrace{\alpha_1\alpha_2}_{R_\gamma}$$

or

$$= 1 \cdot \alpha_2\alpha_3 + \underbrace{\alpha_1(\alpha_2+\alpha_3)}_{R}$$

In fact, graph schemes form an <u>operad</u>. The building blocks of mot_G come from sub & quotient graphs.

<u>Now</u>: $H^q\left(B_I \setminus (Y_G \cap B_I)\right)$ has weights $q \leq w \leq 2q$.

<u>Therefore</u>, a motivic period in RHS of coaction of weight n comes from $H^k(\text{Products of } P_{\gamma_i} \setminus Y_{\gamma_i})$ for $k \leq n$.

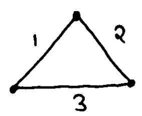
total of $k+1$ edges

Example of small graphs principle: How can we get a $\log^{m}(2)$ in RHS

Only need to look at $\underset{(\checkmark)}{H^1}$, $H^2(\mathbb{P}^2 \setminus Y_\gamma)$ or $\underset{(\text{trivial})}{H^2(\text{products})}$.

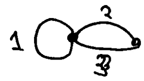
Classify: 3-edge graphs

γ :

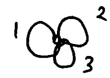


$\alpha_1 + \alpha_2 + \alpha_3$

$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$



$\alpha_1(\alpha_2 + \alpha_3)$



$\alpha_1\alpha_2\alpha_3$

& compute mot_γ. Easy to check unramified $\Rightarrow$ no $\log^{m}(p)$ p prime

Corollary 1: Let $G \in \phi^4$ prim. div. such that $\mathrm{mot}(G) \in MT(\mathbb{Z}[\tfrac{1}{2}])$. Then image of $I^{\mathfrak{m}}_G$ in $\mathbb{Q}\langle g_2, f_3, f_5, \ldots \rangle \otimes \mathbb{Q}[\zeta_2]$ never ends in a g_2.

Eg: In wt 5, & modulo ζ_2 for now, only 2 out of 5 periods

f_5, $\cancel{f_3 g_2^2}$, $\cancel{g_2 f_3}\, g_2$, $g_2^2 f_3$, $\cancel{g_2^5}$

can occur.

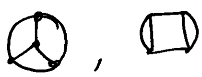
Next: Weight ≤ 2 elements in RHS. Look at H^4, and 5-edge graphs:

etc

Claim: No $\mathrm{Li}_2(\zeta_6)$

Corollary 2: Let $G \in \phi^4$ st $\mathrm{mot}(G) \in MT(\mathbb{Q}(6^{\text{th}}\text{ roots of } 1))$. Then $I^{\mathfrak{m}}_G$ never ends in an f_2.

Examples: Graphs $P_{7,11}$ in wt 11, $P_{8,33}$ in wt 13 (Schnetz, Panzer

Project: Analyse motives of all 6-edge graphs: , etc

Hope to prove (eg. using methods of Dupont) that

$$\mathrm{gr}^{W}_{4}\, \mathrm{mot}_\gamma = 0 .$$

If so, then there is no $\zeta^{\mathfrak{m}}(2)$ in ϕ^4 & so by coaction, there is no $\zeta^{\mathfrak{m}}(n_1, \ldots, n_r)\, \zeta^{\mathfrak{m}}(2)$, for example, at all loop order